Lecture 32

Divergence Theorem

Let D be a simple region whose surface, Σ, is oriented by the normal, $\bar{n}$, directed outward from D, and let $\bar{F}$ be a vector field whose component functions have continuous partial derivatives on D. Then,

$$\iint_{\Sigma} \bar{F}\cdot\bar{n}\, dS = \iiint_{D} \operatorname{div}(\bar{F})\, dV$$

This is a higher dimensional analogue of Green's Theorem. Physically, if $\bar{F}$ represents a velocity field, the LHS represents the rate of fluid flow across the surface Σ. The RHS is a measure of the source/sink nature of the region D.

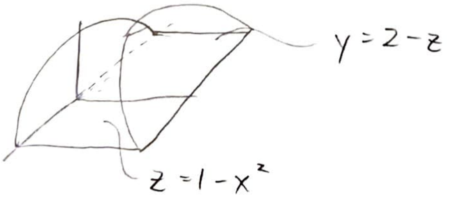
Ex. 1 Find the flux of $\bar{F} = \langle z, y, x \rangle$ over the unit sphere $x^2+y^2+z^2 = 1$.

$$\iint_{\Sigma} \bar{F}\cdot d\bar{S} = \iiint_{D} \operatorname{div}(\bar{F})\, dV$$

$$\operatorname{div}(F) = \nabla\cdot F = \emptyset + 1 + \emptyset = 1$$

$$\iiint 1\, dV = V(B) = \frac{4}{3}\pi$$

Ex. 2 Evaluate $\iint_{\Sigma} \bar{F}\cdot d\bar{S}$ where $\bar{F} = \langle xy,\ y^2+e^{xz^2},\ \sin(xy) \rangle$ and Σ is the surface of the region D bounded by the parabolic cylinder $z = 1-x^2$ and the planes $z=\emptyset$, $y=\emptyset$, & $y+z=2$.



$$\operatorname{div}(\bar{F}) = \frac{\partial}{\partial x} xy + \frac{\partial}{\partial y}\left(y^2+e^{xz^2}\right) + \frac{\partial}{\partial z}\sin(xy) = y+2y$$

$$= 3y$$

$$D = \{(x,y,z) \mid -1 \le x \le 1,\ \emptyset \le z \le 1-x^2,\ \emptyset \le y \le 2-z\}$$

$$\iint_{\Sigma} \bar{F} \cdot d\bar{S} = \iiint_{D} \operatorname{div}(\bar{F})\, dV = \iiint_{D} 3y\, dV$$

$$= 3 \int_{-1}^{1} \int_{\emptyset}^{1-x^2} \int_{\emptyset}^{2-z} y\, dy\, dz\, dx$$

$$= 3 \int_{-1}^{1} \int_{\emptyset}^{1-x^2} \frac{(2-z)^2}{2}\, dz\, dx$$

$$= \frac{3}{2} \int_{-1}^{1} \left(-\frac{(2-z)}{3} \right) \Bigg|_{\emptyset}^{1-x^2} dx$$

$$= -\frac{1}{2} \int_{-1}^{1} \left((x^2+1)^3 - 8 \right) dx$$

$$= -\frac{1}{2} \int_{-1}^{1} x^6 + 3x^4 + 3x^2 - 7\, dx = \frac{184}{35}$$